Lecture 15 : Improper Integrals

In this section, we will extend the concept of the definite integral $\int_a^b f(x)dx$ to functions with an infinite discontinuity and to infinite intervals.

That is integrals of the type

$$\int_1^\infty \frac{1}{x^3}dx \qquad \int_0^1 \frac{1}{x^3}dx \qquad \int_{-\infty}^\infty \frac{1}{4+x^2}$$

Infinite Intervals

An Improper Integral of Type 1

(a) If $\int_a^t f(x)dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

provided that limit exists and is finite.

(c) If $\int_t^b f(x)dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$$

provided that limit exists and is finite.

The improper integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are called **Convergent** if the corresponding limit exists and is finite and **divergent** if the limit does not exists.

(c) If (for any value of a) both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$$

If $f(x) \geq 0$, we can give the definite integral above an area interpretation.

Example Determine whether the integrals $\int_1^\infty \frac{1}{x}dx$, and $\int_{-\infty}^0 e^x dx$ converge or diverge.

Example Determine whether the following integral converges or diverges and if it converges find its value

$$\int_{-\infty}^{\infty} \frac{1}{4+x^2} dx$$

Theorem

$$\boxed{\int_1^{\infty} \frac{1}{x^p} dx \quad \text{is convergent if } p > 1 \text{ and divergent if } p \leq 1}$$

Functions with infinite discontinuities

Improper integrals of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx$$

if that limit exists and is finite.

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$$

if that limit exists and is finite.

The improper integral $\int_a^b f(x)dx$ is called **convergent** if the corresponding limit exists and **Divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Example Determine whether the following integral converges or diverges

$$\int_0^2 \frac{1}{x-2}dx$$

Theorem It is not difficult to show that

$$\boxed{\int_0^1 \frac{1}{x^p}dx \quad \text{is divergent if } p \geq 1 \text{ and convergent if } p < 1}$$

Example determine if the following integral converges or diverges and if it converges find its value.

$$\int_0^4 \frac{1}{(x-2)^2} dx$$

Comparison Test for Integrals

Theorem If f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$, then

(a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.

(b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

Example Use the comparison test to determine if the following integrals are convergent or divergent (using your knowledge of integrals previously calculated).

$$\int_1^\infty \frac{1}{x^2+x+1} dx \qquad \int_1^\infty \frac{1}{x-\frac{1}{2}} dx \qquad \int_0^\pi \frac{\cos^2 x}{\sqrt{x}} dx \qquad \int_0^\infty \frac{e^{-x}}{1+\sin^2 x} dx$$

We have

$$\frac{1}{x^2+x+1} \leq \frac{1}{x^2} \quad \text{if} \quad x > 1,$$

there fore using $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{x^2+x+1}$ in the comparison test above, we can conlude that

$$\int_1^\infty \frac{1}{x^2+x+1} dx$$

converges since

$$\int_1^\infty \frac{1}{x^2} dx$$

converges.

We have

$$\frac{1}{x-\frac{1}{2}} \geq \frac{1}{x} \quad \text{if} \quad x > 1,$$

therfore using

$$f(x) = \frac{1}{x-\frac{1}{2}} \quad \text{and} \quad g(x) = \frac{1}{x}$$

in the comparison test, we have

$$\int_1^\infty \frac{1}{x-\frac{1}{2}} dx$$

diverges since

$$\int_1^\infty \frac{1}{x} dx$$

diverges.

We have

$$\frac{\cos^2 x}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$$

hence using $f(x) = \frac{1}{\sqrt{x}}$ and $g(x) = \frac{\cos^2 x}{\sqrt{x}}$ in the comparison test, we have

$$\int_0^\pi \frac{\cos^2 x}{\sqrt{x}} dx$$

converges, since

$$\int_0^\pi \frac{1}{\sqrt{x}} dx$$

converges.

We have

$$\frac{e^{-x}}{1 + \sin^2 x} \leq e^{-x}$$

Using

$$f(x) = e^{-x}, \quad g(x) = \frac{e^{-x}}{1 + \sin^2 x}$$

in the comparison test, we get

$$\int_0^\infty \frac{e^{-x}}{1 + \sin^2 x} dx$$

converges, since

$$\int_0^\infty e^{-x} dx$$

converges.

Extra Example The standard normal probability distribution has the following formula:

$$\frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$$

The graph is a bell shaped curve. The area beneath this curve is 1 and it fits well to many histograms from data collected. It is used extensively in probability and statistics. to calculate the integral

$$\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$$

you need multivariable calculus. However we can see it the integral converges using the comparison test.

$$e^{\frac{-x^2}{2}} \leq e^{-x/2}$$

when $x \geq 1$. Use this to show that $\int_1^\infty e^{\frac{-x^2}{2}}$ converges.